Grade 8 Math Test Sample Problems

Question 1 John and Peter are having a race on a track of length L meters. John gives Peter a 50-meter head start. They both start at the same time and each runs at a constant speed. John can run at x times Peter's speed, $x > 1$. If John does not win, then

(a) $L \leq \frac{50}{x-1}$ (b) $L \leq \frac{50x}{x+1}$ $\boxed{\text{(c)}}$ $L \leq \frac{50x}{x-1}$ (d) $L \geq \frac{50}{2x-1}$
(e) none of these

Solution

Let v denote Peter's speed. Then John's speed is xv. Peter will take $\frac{L-50}{v}$ seconds to reach the end of the track and John will do so in $\frac{L}{xv}$ seconds. If John does NOT win, then

$$\frac{L-50}{v} \leq \frac{L}{xv}.$$

Thus, $Lx - 50x \leq L$ s0 that $L(x-1) \leq 50x$. Because $x > 1$, we obtain $L \leq \frac{50x}{x-1}$.

Question 2 When the number $4^{78} \times 5^{150}$ is written out, how many digits does it contain?

(a) 151 $\boxed{\text{(b)}}$ 152 (c) 153 (d) 156 (e) None of these

Solution

$$4^{78} \times 5^{150} = 2^{156}5^{150} = 2^6 2^{150} 5^{150} = 2^6 \times 10^{150} = 64 \times 10^{150}.$$

So, the product is 64 followed by 150 zeros.

Question 3 If $1 \leq x \leq 5$, what is the minimum value of $|x-1|+|x-2|+|x-5|$?

(a) 5 (b) 2 (c) 1 $\boxed{\text{(d)}}$ 4 (e) none of these

Solution

Let $1 \leq x \leq 5$. Then $|x-1| + |x-5| = |x-1| + |5-x| = (x-1) + (5-x) = 4$. So, $|x-1| + |x-2| + |5-x| = 4 + |x-2| \geq 4$, with equality when $x = 2$.

Question 4 The length of a side of an equilateral triangle is $\frac{\sqrt{3}+2}{\sqrt{6}}$ centimeters. A square is inscribed in this triangle. In centimeters, the length of the diagonal of this square is

(a) $\sqrt{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\sqrt{3} - 1$ (d) 1 (e) none of these

Question 5 The sum of the first n positive integers is x and the sum of the first $3n$ positive integers is y. The sum of the first $4n$ positive integers is

$\boxed{\text{(a)}}$ $2(y-x)$ (b) $y+2x$ (c) $y+x$ (d) $2y-x$
(e) none of these

Solution

We have

$$1+\cdots+4n=[1+\cdots+n]+\underbrace{[(n+1)+\cdots+3n]}_{y-x}+[(3n+1)+\cdots+4n]$$

Now, Note that

$$\begin{array}{rcl} 3n+1 & = & (2n+1)+n \\ 3n+2 & = & (2n+2)+n \\ & \vdots & \\ 4n & = & (3n)+n. \end{array}$$

Thus,

$$\begin{array}{rl} & [1+\cdots+n]+[(3n+1)+\cdots+4n] \\ = & [1+\cdots+n]+[(2n+1)+\cdots+(3n)]+[n+\cdots+n] \\ = & [1+\cdots+n]+[n+\cdots+n]+[(2n+1)+\cdots+(3n)] \\ = & [(n+1)+(n+2)+\cdots+(2n)]+[(2n+1)+\cdots+(3n)] \\ = & y-x. \end{array}$$

Hence, the desired sum is $2(y-x)$.

Question 6 Let $P(x) = ax^3 + bx^2 + cx + d$. Assume that the remainders when $P(x)$ is divided by $x - 1, x - 2$ and $x - 3$ are $1, 2$, and 3, respectively. What is the remainder when $P(x)$ is divided by $x - 4$?

$\boxed{\text{(a)}}$ $6a + 4$ (b) 4 (c) $a - 4$ (d) 13 (e) none of these

Solution

We have

$$\begin{aligned} 1 = P(1) &= a + b + c + d && (1) \\ 2 = P(2) &= 8a + 4b + 2c + d && (2) \\ 3 = P(3) &= 27a + 9b + 3c + d. && (3) \end{aligned}$$

So, (2)-(1) and (3)-(2) give

$$\begin{aligned} 1 &= 7a + 3b + c && (4) \\ 1 &= 19a + 5b + c. && (5) \end{aligned}$$

Thus, (5)-(4) gives

$$0 = 12a + 2b. \quad (6)$$

That is, $b = -6a$. Using (4), we get $c = 1 + 11a$. Then by (1), we have $d = -6a$. Hence, the remainder when $P(x)$ is divided by $x - 4$ is

$$\begin{aligned} P(4) &= 64a + 16b + 4c + d \\ &= 64a + 16(-6a) + 4(1 + 11a) + (-6a) \\ &= 6a + 4. \end{aligned}$$

Question 7 How many integers b are there for which the parabola $y = x^2 + bx + 4$ does not intersect the line $y = 3x + 1$?

(a) 8 $\boxed{\text{(b)}}$ 7 (c) 6 (d) infinitely many (e) none of these

Solution

The parabola $y = x^2 + bx + 4$ does not intersect the line $y = 3x + 1$ if and only if the equation $x^2 + bx + 4 = 3x + 1$ has no real solutions. This is equivalent to saying that $x^2 + x(b-3) + 3 = 0$ has no real solutions. That is, $(b-3)^2 - 12 < 0$. Hence,

$$|b - 3| < 2\sqrt{3}.$$

There are exactly 7 integers u with $|u| < 2\sqrt{3}$. Hence, there are exactly 7 such integers b.

Question 8 In rectangle $ABCD$, the length of AB is $\sqrt{6}$ centimeters and the length of BC is $\sqrt{10}$ centimeters. F is the midpoint of AC and G is the midpoint of BF. In centimeters, the length of AG is

(a) 2 (b) $\frac{2}{3}\sqrt{6}$ (c) $\frac{8\sqrt{6}+3\sqrt{10}}{12}$ (d) $\frac{\sqrt{15}}{2}$ (e) none of these

Question 9 A *divisor* of an integer n is an integer d such that $\frac{n}{d}$ is also an integer. The number of positive divisors of 800 is

(a) 10 (b) 12 (c) 15 $\boxed{\text{(d)}}$ 18 (e) none of these

Solution

We have $800 = 2^5 \times 5^2$. So, d is a positive divisor of 800 if and only if $d = 2^a 5^b$ for some integers a and b, where $0 \leq a \leq 5, 0 \leq b \leq 2$. So, a may take on 6 different values and b can take on 3 different values. Hence, there are 18 such integers d.

Question 10 Let a, b, and c be distinct real numbers. Consider the equation

$$(x-b)(x-c)+(x-a)(x-c)+(x-a)(x-b)=0.$$

The total number of distinct real solutions that the above equation has is

(a) 0 (b) 1 $\boxed{\text{(c)}}$ 2 (d) dependent on the values of a, b and c
(e) either 0 or 1

Solution

Let $f(x) = (x-b)(x-c)+(x-a)(x-c)+(x-a)(x-b)$. Without loss of generality, we may assume that $a < b < c$. Then $f(b) = (b-a)(b-c) < 0$. Hence, because the graph of $y = f(x)$ is a parabola that opens up, it must intersect the x axis at two distinct points. That is, the equation $f(x) = 0$ has two distinct real solutions.

Question 11 What is the remainder when $2^{2020} + 11$ is divided by 17?

$\boxed{\text{(a)}}$ 10 (b) 12 (c) 13 (d) 4 (e) none of these

Solution

Observe that $2^4 = 17 - 1$ so $2^{2000} = (17-1)^{500} = A - 1$, where A is a multiple of 17. Hence, $2^{2000} + 11 = A + 10$. Since A is a multiple of 17, the remainder of $A + 10$ when divided by 17 is 10.

Question 12 The number

$$\sqrt{4 - 2\sqrt{3}} + \sqrt{4 + 2\sqrt{3}} + \sqrt{2}$$

is a solution to which of the following equations?

(a) $x^4 - 28x^2 + 172 = 0$ (b) $x^4 - 14x^2 - 100 = 0$ (c) $x^4 - 14x^2 + 100 = 0$
$\boxed{\text{(d)}}$ $x^4 - 28x^2 + 100 = 0$ (e) none of these

Solution

Put $z := t + \sqrt{2}$, where $t := \sqrt{4 - 2\sqrt{3}} + \sqrt{4 + 2\sqrt{3}}$. Then

$$t^2 = (4 - 2\sqrt{3}) + (4 + 2\sqrt{3}) + 2\sqrt{16 - 12} = 12.$$

Since $t > 0$, we get $t = 2\sqrt{3}$. Hence, $z^2 = t^2 + 2 + 2t\sqrt{2} = 12 + 2 + 4\sqrt{6}$. Therefore, $(z^2 - 14)^2 = 96$. That is,

$$z^4 - 28z^2 + 100 = 0.$$

Question 13 Suppose that $\sin x - \cos x = \frac{1}{\sqrt{6}}$. Then

$$(\sin^4 x - \cos^4 x)^2 =$$

(a) $\frac{13}{36}$ (b) $\frac{13}{6\sqrt{6}}$ $\boxed{\text{(c)}}$ $\frac{11}{36}$ (d) impossible to determine
(e) none of these

Solution

We will be using the following identities:

$$\sin^2 u + \cos^2 u = 1, \quad \cos(2u) = \cos^2 u - \sin^2 u, \quad 2 \sin u \cos u = \sin(2u).$$

Since $\sin x - \cos x = \frac{1}{\sqrt{6}}$, by squaring both sides, we obtain

$$1 - \sin(2x) = \frac{1}{6}.$$

Thus, $\sin(2x) = 5/6$. Hence,

$$\sin^4 x - \cos^4 x = (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) = -\cos(2x)$$

so that $(\sin^4 x - \cos^4 x)^2 = \cos^2(2x) = 1 - \sin^2(2x) = 1 - \frac{25}{36} = \frac{11}{36}$.

Question 14 Suppose that $x > 0$ and the area of the triangle with vertices $(-1, 0), (2, 4)$ and $(x, 2x - 1)$ is $\frac{49}{4}$ square units. Then $x =$

(a) 3 (b) $\frac{3}{2}$ (c) 9/2 (d) 4 [(e)] none of these

Solution

We use vectors to solve this problem. Let A, B and C denote the points $(-1, 0), (2, 4)$ and $(x, 2x - 1)$ respectively. Let u denote the area of $\triangle ABC$. Then $u = (1/2)(AC)(AB) \sin \angle CAB$. Thus,

$$\begin{aligned}
4u^2 = (AC)^2(AB)^2 \sin^2 \angle CAB &= (AC)^2(AB)^2(1 - \cos^2 \angle CAB) \\
&= (AC)^2(AB)^2 - [\vec{AC} \cdot \vec{AB}]^2 \\
&= [(x+1)^2 + (2x-1)^2][25] - [(x+1, 2x-1) \cdot (3, 4)]^2 \\
&= 25(5x^2 - 2x + 2) - (11x - 1)^2 \\
&= 4x^2 - 28x + 49.
\end{aligned}$$

With $u = \frac{49}{4}$, the correct choice is "none of these".

Question 15 Let $x \geq 1$. A piece of string is cut in two at a point selected at random. The probability that the longer piece is at least x times as long as the shorter piece is

(a) $\frac{1}{2}$ (b) $\frac{1}{2x-1}$ (c) $\frac{1}{x+1}$ (d) $\frac{1}{x}$ [(e)] $\frac{2}{x+1}$

Solution

Suppose the length of the string is L. We may rephrase this problem as follows: A number t in the interval $I := [0, L]$ is chosen at random so that I is divided into two subintervals. What is the probability that the longer interval has length at least x times the length of the shorter interval?

So, suppose t is a number in I. Then the two subintervals of I are $I_1 := [0, t]$ and $I_2 := [t, L]$. We want to determine all values of t so that either the length of I_2 is at least x times the length of I_1 or the length of I_1 is at least x times the length of I_2. So, let us determine all values of t so that either $L - t \geq xt$ or $t \geq x(L - t)$. Solving these inequalities, we get

$$t \in \left[0, \frac{L}{x+1}\right] \quad \text{or} \quad t \in \left[\frac{xL}{1+x}, L\right].$$

So, the desired probability is

$$\frac{\frac{L}{x+1} + \left(L - \frac{xL}{1+x}\right)}{L} = \frac{1}{x+1} + \left(1 - \frac{x}{1+x}\right) = \frac{2}{1+x}.$$

Question 16 When $\left(x + \frac{1}{x^2}\right)^{50}$ is expanded and simplified, the coefficient of x^{41} is

[(a)] 140^2 (b) $\frac{50!}{41!\,9!}$ (c) $\frac{50!}{23!\,27!}$ (d) 0 (e) none of these

Solution

Recall that for any non-negative integer n, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \quad \text{where } \binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

With $a := x, b := x^{-2}$, we get

$$\left(x + \frac{1}{x^2}\right)^{50} = \sum_{k=0}^{50} \binom{50}{k} x^k (x^{-2})^{50-k} = \sum_{k=0}^{50} \binom{50}{k} x^{3k-100}.$$

Now, $3k - 100 = 41 \iff k = 47$ so that the coefficient of x^{41} is

$$\binom{50}{47} = \frac{50 \times 49 \times 48}{6} = 25 \times 49 \times 16 = (5 \times 7 \times 4)^2 = 140^2.$$

Question 17 There are constants $a_0, a_1, a_2, \ldots, a_{2020}$ such that for all $x \in \mathbb{R}$,

$$(x^2 - 2x)^{1010} = a_0 + a_1(x-1) + a_2(x-1)^2 + \cdots + a_{2020}(x-1)^{2020}.$$

Then $a_2 =$

(a) 2020 (b) 1010 (c) -2020 (d) -505

[(e)] none of these

Solution

Let $g(x) := (x^2 - 2x)^{1010}$. We are given that for all $x \in \mathbb{R}$,

$$g(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + \cdots + a_{2020}(x-1)^{2020}.$$

Hence, taking the second derivative of g gives

$$g''(x) = 2a_2 + 6a_3(x-1) + \cdots + 2020(2019)a_{2020}(x-1)^{2018}.$$

So, $g''(1) = 2a_2$. But since $g(x) = (x^2 - 2x)^{1010}$, direct differentiation yields

$$g'(x) = 1010(x^2 - 2x)^{1009}(2x - 2) = 2020(x-1)(x^2 - 2x)^{1009}.$$

Therefore,

$$g''(x) = 2020[(x^2 - 2x)^{1009} + (x-1)(1009)(x^2 - 2x)^{1008}(2x - 2)].$$

Thus, $g''(1) = 2020[-1] = -2020$. Hence, $a_2 = -1010$.

Question 18 Let $f : \mathbb{R} \to \mathbb{R}$ with $\lim_{x\to 0} \frac{f(x)}{x} = 10$. Then

$$\lim_{x\to -2} \frac{f(x^3 + 2x^2 - x - 2)}{x^2 - x - 6} =$$

(a) 10 $\boxed{\text{(b)}}$ -6 (c) -2 (d) 30 (e) none of these

Solution

Put

$$\begin{aligned} u &:= x^3 + 2x^2 - x - 2, \\ v &:= x^2 - x - 6. \end{aligned}$$

Then $v = u\left(\frac{x-3}{x^2-1}\right)$. So,

$$\frac{f(u)}{v} = \left[\frac{f(u)}{u}\right]\left[\frac{x^2-1}{x-3}\right].$$

As $x \to -2$, we have $u \to 0$, and conversely. Hence,

$$\begin{aligned} \lim_{x\to -2} \frac{f(x^3 + 2x^2 - x - 2)}{x^2 - x - 6} &= \lim_{x\to -2} \frac{f(u)}{v} \\ &= \left[\lim_{u\to 0} \frac{f(u)}{u}\right]\left[\lim_{x\to -2} \frac{x^2-1}{x-3}\right] \\ &= 10\left[\frac{-3}{5}\right] = -6. \end{aligned}$$

Question 19 There are two lines that are simultaneously tangent to both the parabola $y = x^2 + 1$ and the parabola $y = -2(x+1)^2 - 3$. The sum of the slopes of these two lines is

$\boxed{\text{(a)}}$ $-\frac{8}{3}$ (b) 6 (c) 8 (d) 20/3 (e) none of these

Solution

Suppose a line ℓ is tangent to the curve $y = f(x) := x^2 + 1$ and the curve $y = g(x) := -2(x+1)^2 - 3$ at $P(a, a^2 + 1)$ and $Q(b, -2(b+1)^2 - 3)$, respectively. Since ℓ is tangent to $y = f(x)$ at P, its slope is $f'(a)$. Similarly, since ℓ is tangent to $y = g(x)$ at Q, the slope of ℓ is $g'(b)$. Therefore,

$$2a = f'(a) = g'(b) = -4(b+1).$$

Thus, $a = -2(b+1)$. On the other hand, the slope of ℓ can be computed directly using P and Q so that

$$\frac{a^2 + 1 + 2(b+1)^2 + 3}{a - b} = -4(b+1).$$

Thus,

$$[4(b+1)^2+1]+[2(b+1)^2+3]=[-4(b+1)][-3b-2].$$

Expand and simplify, we obtain $3b^2+4b-1=0$. Thus,

$$b=\frac{-2\pm\sqrt{7}}{3}.$$

Therefore, the slopes of the two lines that are simultaneously tangent to both the parabola $y=x^2+1$ and the parabola $y=-2(x+1)^2-3$ are

$$-4(b+1)=\frac{-4\pm 4\sqrt{7}}{3}.$$

The sum of these slopes is $-\frac{8}{3}$.

Question 20 Suppose that u is a constant. Let M and m be, respectively, the maximum and the minimum number in the set

$$\left\{\frac{2t^2-t+u}{1+t+t^2}:t\in\mathbb{R}\right\}.$$

Then $3(M+m)=$

(a) $8+4u$ (b) $2+4u$ $\boxed{\text{(c)}}$ $10+4u$ (d) $2u+1$
(e) none of these

Solution

Let

$$f(t):=\frac{2t^2-t+u}{1+t+t^2}.$$

Let R denote the range of f. Then $R=[m,M]$. On the other hand, R consists of all y for which $f(t)=y$ for some real t. That is, $y\in R$ if and only if there is some real t satisfying

$$t^2(2-y)+t(-1-y)+(u-y)=0. \tag{7}$$

Thus, $y\in R$ if and only if $(1+y)^2-4(2-y)(u-y)\geq 0$, i.e.,

$$3y^2+y(-10-4u)+(8u-1)\leq 0. \tag{8}$$

The solution set of (8) is $[r_1,r_2]$, where $r_1<r_2$ are the two real roots of $3y^2+y(-10-4u)+(8u-1)=0$. Hence, the range of f is $[r_1,r_2]$. But then $r_1=m, r_2=M$ so that

$$m+M=r_1+r_2=\frac{10+4u}{3}.$$

Therefore, $3(m+M) = 10 + 4u$.

(Note: Equation (7) clearly has a solution in t if $y = 2$. So, $2 \in R$. To derive the condition (8), we assumed that (7) is a quadratic equation, i.e, we implicitly assumed that $y \neq 2$. So, strictly speaking, $R = \{2\} \cup ([r_1, r_2] \setminus \{2\})$. However, it is obvious that $y = 2$ also a solution to inequality (8) for all values of u, Therefore, $[r_1, r_2]$ *already* contains 2. Hence, $R = \{2\} \cup ([r_1, r_2] \setminus \{2\}) = [r_1, r_2]$.)